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PORTFOLIO CHOICE PROBLEM WITH THE VALUE-AT-RISK UTILITY FUNCTION UNDER GENERAL LINEAR CONSTRAINTS

1. Introduction.

The method of optimal portfolio construction was developed by Markowitz in 1952 [1]. This method is a standard basis for optimal risk assets portfolio weights (structure) calculation till now. Portfolio constructed by this method has the smallest risk for selected level of expected return. Changing the level of portfolio expected return we can construct the set of optimal portfolios which is known as an efficient frontier. It can be easily shown that using the portfolio variance as a risk measure this set is a parabola and in the case of portfolio standard deviation as a risk measure – a hyperbola [2]. The main property of an efficient frontier is impossibility of portfolio expected return increasing without increasing portfolio risk (variance) or equivalent impossibility of portfolio risk (variance) decreasing without decreasing portfolio expected return.

In financial literature there are some other methods of optimal portfolio construction. The special case is maximization of portfolio utility [3]. Maximizing the investor's utility we get the optimal portfolio which also lies on efficient frontier. It should be noted that this portfolio depends on investors risk aversion. In the case when investor is fully risk avers the maximum expected quadratic utility portfolio coincide with minimum variance portfolio. Changing the coefficient of investor's risk aversion from 0 to we get the efficient frontier. That is maximum expected quadratic utility portfolio is generalization of portfolio theory.

In known methods of portfolio construction the portfolio variance is taken as a risk measure. Such an approach is heavily criticized during last decades. First of all it is caused by the fact that variance gives information only about dispersion of possible values of portfolio return around portfolio expected return but not about the portfolio risk. Moreover, portfolio variance takes into account two-sided risk. It means that high portfolio returns probability increasing leads to portfolio variance increasing which signals investor about portfolio risk increasing. But in fact portfolio risk should not increase in this situation. Better instruments for portfolio risk describing obviously are functions which takes into account only positive values of portfolio loss function (or equivalent only negative values of portfolio return) and are more informative than portfolio variance.

In the last years of previous century the investigations provided in risk theory showed that the quantile based risk measures can be useful for practice. The simplest and the most known such a measure is Value-at-Risk (henceforth VaR). This measure is recommended by Basle Committee [4]. The conception of the VaR was first proposed in [5]. Thanks to results which can be easily interpreted this instrument for risk calculation is nowadays the most popular in finance and econometrics. Taking this into account it is proposed in [6] to use the VaR as a main risk measure in Markowitz's analysis. Assuming that asset returns are normally distributed in [6] the analytical solution of the portfolio VaR minimization problem is found and it is shown that minimum VaR portfolio lies on efficient frontier but has higher expected return and consequently higher variance, than the minimum variance portfolio.

The natural question is: is it possible to use the conception of portfolio utility maximization for its construction with the VaR as a risk measure?

In the paper it is proposed to construct the portfolio on the basis of utility maximization with the VaR as a tool for risk calculation. This portfolio better fits the recommendations of Basle Committee than the portfolio with maximum utility with portfolio variance as a risk measure. It allows the banks to provide the operations on fund market more intensively in the Basle Committee and law bounds. Moreover, as it was pointed out, the VaR approach to risk calculation is more correct than the variance which allows more precise consideration of financial risk in process of portfolio construction.

2. Markowitz's optimization problem with general linear restrictions.

Denote by P_t – the price of asset at time point t, and define an asset return at this time point as:

$$X_t = 100\ln\frac{P_t}{P_{t-1}}.$$

Note that in financial mathematics literature asset returns are mostly used for calculation because their properties are more statistically attractive than properties of asset's price. The asset returns are unbounded which is one of its main advantages. Moreover, asset returns have no time trend and their values are dissipated around zero.

Behavior of asset returns has random nature. That's why it is often assumed that asset returns are random variables. Let we construct a portfolio with k assets. Denote $X_t = (X_{1t}, X_{2t}, ..., X_{kt})$ the k-dimensional vector of asset returns. The vector $w = (w_1, w_2, ..., w_k)$ stands for portfolio where w_i – the fraction of investor's wealth invested into i-th asset. We assume that X_t is k-dimensional normally distributed random variable with parameters μ and Σ. Such assumption is criticized in the last decades because distributions of asset returns are heavy tailed. In [7] it is shown that under good diversification the impact of heavy tails on portfolio characteristics is not essential. The portfolio expected return can be calculated as $Rw = E(X_{wt}) = \mu' w$, portfolio variance $Vw = D(X_{wt}) = \mu' \sum w$, where X_{wt} – portfolio return at time point t.

In the classical portfolio theory expected quadratic utility has the form:

$$U(\mathbf{w}) = R_{\mathrm{w}} - \frac{\beta}{2} V_{\mathrm{w}},$$

where β denotes the coefficient which describes investor's attitude towards risk or in other words investor's risk aversion. It is assumed that this coefficient is known. If investor constructs his portfolio only with risky assets then the problem of rational portfolio construction has the following form

$$U(w) \to \max \text{ with respect to } \sum_{i=1}^{k} w_i = 1 . \tag{1}$$

The solution of problem (1) is the maximum utility portfolio with the following weights:

$$w_{EU} = w_{GMV} + \beta^{-1} R\mu , \tag{2}$$

where $w_{GMV} = \dfrac{\Sigma^{-1} i}{i'\Sigma^{-1} i}$ – the weights of minimum variance portfolio, $i - k$ dimensional vector of ones $R = \Sigma^{-1} - \dfrac{\Sigma^{-1} i i' \Sigma^{-1}}{i'\Sigma^{-1} i}$.

The optimization problem (1) can be generalized to the form:

$$U(w) \to \max \text{ with respect to } A'w = b , \tag{3}$$

where $A - k * q$ matrix ($q \le k$) with rank q, $b - q*1$ vector. The solution of (3) is given in [8]:

$$\mathbf{w}_{EU,\mathbf{A}} = \mathbf{w}_{GMV,\mathbf{A}} + \beta^{-1} \mathbf{R}_{\mathbf{A}} \boldsymbol{\mu} , \tag{4}$$

where

$$\mathbf{R}_{\mathbf{A}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1} \text{ and } \mathbf{w}_{GMV,\mathbf{A}} = \boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{b} .$$

The statistical and probability properties of portfolio weights (2) and (4) are considered in [3], [8]-[11].

Note that portfolio with the weights $\mathbf{w}_{GMV,\mathbf{A}}$ is the solution of Markowitz's classical risk minimization problem with no conditions on portfolio expected return:

$$V_{\mathbf{w}} \to \min \text{ with respect to } \mathbf{A}'\mathbf{w} = \mathbf{b} . \tag{5}$$

The expected return and variance of portfolio $\mathbf{w}_{GMV,\mathbf{A}}$ have the form:

$$R_{GMV,\mathbf{A}} = \boldsymbol{\mu}'\mathbf{w}_{GMV,\mathbf{A}} = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{b} \text{ and } V_{GMV,\mathbf{A}} = \mathbf{w}'_{GMV,\mathbf{A}}\boldsymbol{\Sigma}\mathbf{w}_{GMV,\mathbf{A}} = \mathbf{b}'(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{b} . \tag{6}$$

Consider the classical Markowitz's problem:

$$V_{\mathbf{w}} \to \min \text{ with respect to } \mathbf{A}'\mathbf{w} = \mathbf{b} \text{ and } \mathbf{w}'\boldsymbol{\mu} = R \tag{7}$$

and define the notion of efficient frontier for this problem. Obviously that the necessary condition for (7) to be correct is $R \ge R_{GMV,\mathbf{A}}$. Denote by W the set of all portfolios which consist of k assets which fulfill the condition $\mathbf{A}'\mathbf{w} = \mathbf{b}$.

Definition 1. The subset E of the set W is an efficient frontier for problem (7) if for portfolios which belong to E it is impossible to increase their expected return without increasing their risk (variance) and it is impossible to decrease their risk without decreasing their expected return in the bounds of W.

Lemma 1. For arbitrary real number $R \ge R_{GMV,\mathbf{A}}$ exists single portfolio w_R with expected return R which belongs to an efficient frontier E with the weights:

$$\mathbf{w}_R = \boldsymbol{\Sigma}^{-1}\begin{pmatrix}\boldsymbol{\mu} & \mathbf{A}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A} \\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R \\ \mathbf{b}\end{pmatrix}.$$

Proof. We solve problem (7) using the method of Lagrange multipliers. Let λ_1 be real number and λ_2 – q-dimensional vector. Denote $\lambda = (\lambda_1 \quad \lambda_2')'$. Then the Lagrange function can be written:

$$L(\mathbf{w},\boldsymbol{\lambda}) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \boldsymbol{\lambda}'\left(\begin{pmatrix}\boldsymbol{\mu}'\\ \mathbf{A}'\end{pmatrix}\mathbf{w} - \begin{pmatrix}R\\ \mathbf{b}\end{pmatrix}\right) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \lambda_1(\boldsymbol{\mu}'\mathbf{w} - R) - \boldsymbol{\lambda}_2(\mathbf{A}'\mathbf{w} - \mathbf{b}).$$

We put the partial derivatives of $L(w,\lambda)$ equal to zero:

$$\begin{cases}\dfrac{\partial L}{\partial \mathbf{w}} = 2\boldsymbol{\Sigma}\mathbf{w} - \begin{pmatrix}\boldsymbol{\mu} & \mathbf{A}\end{pmatrix}\boldsymbol{\lambda} = \mathbf{O}_k\\ \dfrac{\partial L}{\partial \boldsymbol{\lambda}} = \begin{pmatrix}\boldsymbol{\mu}'\\ \mathbf{A}'\end{pmatrix}\mathbf{w} - \begin{pmatrix}R\\ \mathbf{b}\end{pmatrix} = \mathbf{O}_{q+1}\end{cases}, \tag{8}$$

where O_k and O_{q+1} k and $q+1$-dimensional zero vectors.

From the first equation of (8) we get

$$\mathbf{w} = \frac{1}{2}\boldsymbol{\Sigma}^{-1}\begin{pmatrix}\boldsymbol{\mu} & \mathbf{A}\end{pmatrix}\boldsymbol{\lambda}.$$

Substitute previous result in the second equation of (8) and solve it with respect to λ we observe:

$$\boldsymbol{\lambda} = 2\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R\\ \mathbf{b}\end{pmatrix}.$$

Hence we get:

$$\begin{cases}\boldsymbol{\lambda} = 2\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R\\ \mathbf{b}\end{pmatrix}\\ \mathbf{w}_R = \boldsymbol{\Sigma}^{-1}\begin{pmatrix}\boldsymbol{\mu} & \mathbf{A}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R\\ \mathbf{b}\end{pmatrix}\end{cases},$$

which proves lemma 1.

Lemma 2. Let portfolio w_R with expected return R_w and variance V_w belongs to efficient frontier E, then:

$$\left(R_{\mathbf{w}} - R_{GMV,\mathbf{A}}\right)^2 = s_{\mathbf{A}}\left(V_{\mathbf{w}} - V_{GMV,\mathbf{A}}\right), \tag{9}$$

where $s_{\mathbf{A}} = \boldsymbol{\mu}'\mathbf{R}_{\mathbf{A}}\boldsymbol{\mu}$.

Proof. Consider the following block matrix

$$\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1} = \begin{pmatrix}b_{11} & \mathbf{B}_{12}\\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{pmatrix}.$$

Using the rules of constructing inverse block matrices we get:

$$b_{11} = \left(\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}'\right)^{-1} = \left(\boldsymbol{\mu}'\mathbf{R}_{\mathbf{A}}\boldsymbol{\mu}\right)^{-1} = s_{\mathbf{A}}^{-1},$$

$$\mathbf{B}_{21} = -b_{11}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}',\ \mathbf{B}_{12} = \mathbf{B}_{21}^{'},\ \mathbf{B}_{22} = \frac{\mathbf{B}_{21}\mathbf{B}_{12}}{b_{11}} + (\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}.$$

We can write:

$$V_{\mathbf{w}} = \mathbf{w}_R^{'}\boldsymbol{\Sigma}\mathbf{w}_R = \begin{pmatrix}R_{\mathbf{w}} & \mathbf{b}'\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\mu}'\\ \mathbf{A}'\end{pmatrix}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\begin{pmatrix}\boldsymbol{\mu} & \mathbf{A}\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R_{\mathbf{w}}\\ \mathbf{b}\end{pmatrix} =$$

$$= \begin{pmatrix}R_{\mathbf{w}} & \mathbf{b}'\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\\ \mathbf{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} & \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}\end{pmatrix}^{-1}\begin{pmatrix}R_w\\ \mathbf{b}\end{pmatrix} = b_{11}R_{\mathbf{w}}^2 + 2R_w\mathbf{b}'\mathbf{B}_{21} + \mathbf{b}'\mathbf{B}_{22}\mathbf{b} =$$

$$= b_{11}\left(R_w + \frac{\mathbf{b}'\mathbf{B}_{21}}{b_{11}}\right)^2 + \mathbf{b}'\left(\mathbf{B}_{22} - \frac{\mathbf{B}_{21}\mathbf{B}_{12}}{b_{11}}\right)\mathbf{b}.$$

Hence:

$$\left(R_{\mathbf{w}} - R_{GMV,\mathbf{A}}\right)^2 = s_{\mathbf{A}}\left(V_{\mathbf{w}} - V_{GMV,\mathbf{A}}\right).$$

It should be noted that in mean-variance space equation (9) describes the efficient frontier E.

3. Maximization of utility based on Value-at-Risk.

As it is pointed out previously portfolio variance gives only few information about portfolio risk even in the case of normally distributed asset returns. The variance reflects the dispersion of the possible values of portfolio returns around its expected return. That's why the problem of using utility based on better instruments for risk calculation for portfolio construction arises.

Since the VaR is the most widen risk measure nowadays the utility based on the VaR utilization for portfolio construction should be considered. We examine the following function of expected utility of investor:

$$U_{VaR}(\mathbf{w}) = R_{\mathbf{w}} - \frac{\beta}{2} M_{\mathbf{w}},$$

where M_w – VaR of portfolio w. Note that under assumption of asset returns normality the portfolio VaR can be expressed as $M_{\mathbf{w}} = z_\alpha \sqrt{V_{\mathbf{w}}} - R_{\mathbf{w}}$, where α – confidence level for VaR and $z_\alpha = -\Phi^{-1}(1-\alpha)$ is α-quantile of standard normal distribution. We consider the expected utility maximization problem which is analogical to (3):

$$U_{VaR}(w) \to \max \text{ with respect to } A'w = b. \tag{10}$$

The solution to the problem (10) is given in the next theorem.

Theorem 1. Let we construct a portfolio with k risky assets. Denote X_t – k-dimensional vector of asset returns at time point t. Assume that $X_t \sim N(\mu, \Sigma)$. Then the solution to the maximization problem (10) has the form:

$$\mathbf{w}_{VaR,\mathbf{A}} = \mathbf{w}_{GMV,\mathbf{A}} + \frac{\sqrt{V_{GMV,\mathbf{A}}}}{\sqrt{\tilde{z}_\alpha^2 - s_{\mathbf{A}}}} \mathbf{R}_{\mathbf{A}} \boldsymbol{\mu},$$

where $\tilde{z}_\alpha = \frac{\beta}{\beta+2} z_\alpha$. Moreover the necessary and sufficient condition of solution existence is $\tilde{z}_\alpha^2 > s_{\mathbf{A}}$.

Proof. First we show that if the solution of (10) exists then it belongs to an efficient frontier E. We prove this by contradiction. We assume that there exists a portfolio w which solves (10) (and belongs to W) but does not belong to E. From the definition of efficient frontier there exists a portfolio w_0 such that $R_{\mathbf{w}_0} \geq R_{\mathbf{w}}$, $V_{\mathbf{w}_0} \leq V_{\mathbf{w}}$ and one of the previous inequalities is strict. Then

$$U_{VaR}(w_0) = R_{\mathbf{w}_0} - \frac{\beta}{2}\left(z_\alpha \sqrt{V_{\mathbf{w}_0}} - R_{\mathbf{w}_0}\right) = \left(1 + \frac{\beta}{2}\right) R_{\mathbf{w}_0} - \frac{\beta}{2} z_\alpha \sqrt{V_{\mathbf{w}_0}} > \left(1 + \frac{\beta}{2}\right) R_w - \frac{\beta}{2} z_\alpha \sqrt{V_w} = U_{VaR}(w)$$

Which is a contradiction to our assumption that portfolio w is a solution of problem (10).

Note that (10) is equivalent to the problem

$$R_w - \tilde{z}_\alpha \sqrt{V_w} \to \max, \text{ if } A'w = b, \tag{11}$$

since

$$U_{VaR}(\mathbf{w}) = R_{\mathbf{w}} - \frac{\beta}{2}\left(z_\alpha \sqrt{V_{\mathbf{w}}} - R_{\mathbf{w}}\right) = \left(1 + \frac{\beta}{2}\right) R_{\mathbf{w}} - \frac{\beta}{2} z_\alpha \sqrt{V_{\mathbf{w}}} = \left(1 + \frac{\beta}{2}\right)\left(R_{\mathbf{w}} - \tilde{z}_\alpha \sqrt{V_{\mathbf{w}}}\right).$$

Because the solution of problem (11) belongs to efficient frontier E then the result of lemma 2 holds, namely the relation between expected return and variance (9). We solve (9) with respect to $R_{\mathbf{w}}$:

$$R_{\mathbf{w}} = R_{GMV,\mathbf{A}} + \sqrt{s_{\mathbf{A}}\left(V_{\mathbf{w}} - V_{GMV,\mathbf{A}}\right)}$$

and consider the optimization problem

$$R_{\mathbf{w}} - \tilde{z}_{\alpha}\sqrt{V_{\mathbf{w}}} \rightarrow \max \text{ with respect to } R_{\mathbf{w}} = R_{GMV,\mathbf{A}} + \sqrt{s_{\mathbf{A}}\left(V_{\mathbf{w}} - V_{GMV,\mathbf{A}}\right)}, \qquad (12)$$

which is equivalent to the problem (11) and also it is equivalent to the unconditional maximization of a function $R_{GMV,\mathbf{A}} + \sqrt{s_{\mathbf{A}}\left(V_{\mathbf{w}} - V_{GMV,\mathbf{A}}\right)} - \tilde{z}_{\alpha}\sqrt{V_{\mathbf{w}}}$. It can be easily shown that this function reaches its maximum at the point:

$$V_{\mathbf{w}} = \frac{\tilde{z}_{\alpha}^2}{\tilde{z}_{\alpha}^2 - s_{\mathbf{A}}} V_{GMV,\mathbf{A}}.$$

Hence the portfolio constructed from the optimization problem (10) has the variance $V_{\mathbf{w}} = \frac{\tilde{z}_{\alpha}^2}{\tilde{z}_{\alpha}^2 - s_{\mathbf{A}}} V_{GMV,\mathbf{A}}$. Using the results of lemmas 1 and 2 we get the statement of the theorem.

The necessity and sufficiency of condition $\tilde{z}_{\alpha}^2 > s_{\mathbf{A}}$ can be proved analogically to proposition 1 in [6].

Consider the classical problem of investor's expected utility maximization for portfolio construction

$$U_{VaR}(w) \rightarrow \max \text{ with respect to} \sum_{i=1}^{k} w_i = 1. \qquad (13)$$

Using the results of theorem 1 the solution of problem (13) can be easily found.

Corollary 1. Let we construct a portfolio with k risky assets. Denote X_t – k-dimensional vector of asset returns at time point t. Assume that $X_t \sim N(\mu, \Sigma)$. Then the solution to the maximization problem (13) has the form

$$\mathbf{w}_{VaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{\tilde{z}_{\alpha}^2 - s}} \mathbf{R}\boldsymbol{\mu},$$

where $s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}$. Moreover the necessary and sufficient condition of solution existence is $s < \tilde{z}_{\alpha}^2$.

Proof. Replacing in the expression for weights $wVar$, $A = i$ and $b = 1$, we get the necessary statement.

Remark 1. The assumption of asset returns normality can be essentially weakened. So the results of the paper leave true if we assume that vector X_t has k-dimensional conditional normal distribution with parameters $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$. As a special case we can consider for example the assumption that asset returns follow k-dimensional VARMA-GARCH with normally distributed residuals. Also the assumption of elliptically distributed residuals does not influence the results of the paper but in this case the quantile z_a should be changed by appropriate a-quantile of respective elliptical distribution.

Remark 2. The portfolios with weigts $\mathbf{w}_{EU,\mathbf{A}}$ and $\mathbf{w}_{VaR,\mathbf{A}}$ both belong to the efficient frontier E and maximize respective functions of investor's expected utility. In general case we are not able to put the equality sign between these weights, that is $\mathbf{w}_{EU,\mathbf{A}} \neq \mathbf{w}_{VaR,\mathbf{A}}$. But it always exist β_{EU} β_{VaR} such that the solution of problem (3) with coefficient β_{EU} coincides with the solution of problem (10) with coefficient β_{VaR}.

Remark 3. We can consider the problem (10) with the conditional Value-at-Risk as a proxy for risk calculation. In this case the results of the theorem 1 still true if in formula for $\mathbf{w}_{VaR,\mathbf{A}}$ we replace z_α by $k_\alpha = \frac{-\int_{-\infty}^{-z_\alpha} x\phi(x)dx}{1-\alpha} = \frac{\exp(-\frac{z_\alpha^2}{2})}{\sqrt{2\pi}(1-\alpha)}$.

4. Conclusion.

The paper examines the problem of portfolio of risky assets construction with the maximum expected utility which is based on the Value-at-Risk as a proxy for risk calculation. Contrary to the classical method of expected quadratic utility maximization for portfolio construction considered approach is not examined in scientific works because the concept of VaR for portfolio construction is relatively new.

In the paper we consider the generalized problem of portfolio construction where the classical condition (the sum of portfolio weights is equal to one) is replaced by q linear restrictions on portfolio weights. We construct the efficient frontier for this problem and formulate the necessary condition for portfolio characteristics which should be satisfied by portfolios which belong to efficient frontier. As a corollary from theorem 1 we get the solution to the portfolio optimization problem with the classical restrictions.

The utilization of the described method of portfolio construction especially in banking is fully agreed with the recommendations of Basle Committee. As a consequence this method gives the banks possibility to provide the operations on the fund market in the bounds of Basle agreement. Moreover the competent establishment of conditions gives the possibility to take into account all standards and restrictions provided by existing law.

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